A two-dimensional singular function via Sturmian words in base \( \beta \)

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Abstract

For \( \beta > 1 \), a function \( (\cdot)_\beta \) maps each infinite word \( a_1a_2\cdots \in \mathbb{N}^\infty \) to a real number \( \sum_{i=1}^\infty a_i/\beta^i \). We define \( \Xi(\alpha, \beta) \) by \( (s'_\alpha,0)_\beta \) where \( s'_\alpha,0 \) is a lexicographically greatest mechanical word of slope \( \alpha \). This paper demonstrates that the function \( \Xi \) enjoys devil’s staircase-like properties. Its continuity, partial and total differentiability will be investigated. We also present a set of \( \Xi \)-values, in which any finite number of members are algebraically independent over the field of rationals.

Keywords: \( \beta \)-shift, singular function, Sturmian word, irrationality measure, Liouville number

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1. Introduction

Throughout the paper, we denote by \( \lfloor \cdot \rfloor \) (resp. \( \lceil \cdot \rceil \)) the usual floor (resp. ceiling) function, and by \( \{t\} \) the fractional part of a real \( t \), i.e., \( t = \lfloor t \rfloor + \{t\} \). Let \( \alpha \geq 0 \) and \( 0 \leq \rho \leq 1 \). We define, for integers \( n \geq 0 \),

\[
s_{\alpha,\rho}(n) := \lfloor \alpha(n + 1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,
\]

\[
s'_{\alpha,\rho}(n) := \lceil \alpha(n + 1) + \rho \rceil - \lceil \alpha n + \rho \rceil,
\]

and consider two infinite words \( s_{\alpha,\rho} = s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots \) and \( s'_{\alpha,\rho} = s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots \) called lower and upper mechanical word respectively with slope \( \alpha \) and intercept \( \rho \). Mechanical words with irrational slopes are also called Sturmian.
words. Note that $s_{α,ρ}(n)$ and $s'_{α,ρ}(n)$ are equal to either $[α] − 1$ or $[α]$ for each $n ≥ 0$. In particular, $s_{α,ρ} = s'_{α,ρ} = αω := ααα⋯$ if and only if $α$ is an integer. For a general survey on mechanical words, see [14].

For $β > 1$, a function $(·)_β$ sends each infinite word $a_1a_2⋯ ∈ \mathbb{N}^\infty$ to a real number $\sum_{i=1}^{∞} a_i/β^i$. Now our primary concern is a two-variable function $Ξ : (0, ∞) × (1, ∞) → \mathbb{R}$ which is defined by

$$
Ξ(α, β) := (s'_{α,0})_β.
$$

In contrast, the function $∆ : [0, ∞) → \mathbb{R}$ considered in [10] is defined so that $∆(α) ≥ 1$ is a real solution to a series equation

$$
1 = (s'_{α,0})_x = \sum_{n=1}^{∞} s'_{α,0}(n − 1)/x^n.
$$

The author showed in [12] that, via invariant orbits of $β$-transformations with frequency $α$, the map $Ξ$ is realized as long as $∆(α) ≤ β$.

This paper studies the continuity and the partial and total differentiability of $Ξ$, and as a consequence, demonstrates that $Ξ$ has interesting properties in common with devil’s staircases (= singular functions). Doing so, there is no need to confine ourselves to the case $∆(α) ≤ β$. Thus $(α, β)$ will be assumed to be any pair in $(0, ∞) × (1, ∞)$. We will see that the function $Ξ$ simultaneously generalizes the former devil’s staircases in the literature. The left of Figure 1 represents the graph of $z = Ξ(α, β)$, whereas on the right we impose an additional restriction $∆(α) ≤ β$. Discontinuous jumps are observed wherever $α$ is rational. On the top face of the right (i.e., the level curve $Ξ(α, β) = 1$), one recognizes the devil’s staircase of [10, Figure 1]. Similarly, we can find, on the section of $β = 2$, the devil’s staircase of [4, Figure 1] as a mirror image with respect to the line $β = α$. Accordingly, we get a 3-dimensional picture of both studies [4, 10]. Recall, in addition, that the level curve $Ξ(α, β) = 1$ completely characterized generalized baker’s transformations [11]. On the other hand, a measure-theoretical study of $µ_β(x) := Ξ(x, β)$ was pursued in [13]. In particular, the moments associated with $µ_β(x)$ were thoroughly investigated.

Furthermore, a two-dimensional singular function is obtained from $Ξ$. Observing Figure 1, we can define a function $Θ : (1, ∞) × (0, ∞) → \mathbb{R}$ in such a way that $α = Θ(β, Ξ)$. This point of view was already adopted and carefully studied in [4]. Then this two-variable function $Θ$ accomplishes
the classical definition of a singular function, that is to say, a non-constant function which is continuous, monotonic and locally constant on a set of full measure.

Instead of our \( \Xi(\alpha, \beta) := (s'_{\alpha,0})_\beta \), it seems quite natural to consider two-variable function \( (s_{\alpha,0})_\beta \), or even more generally, \( (s_{\alpha,\rho})_\beta \) and \( (s'_{\alpha,\rho})_\beta \) with \( \rho \neq 0 \) fixed. One may recognize that the analysis below for \( (s'_{\alpha,0})_\beta \) also works for \( (s_{\alpha,0})_\beta \) with little modification. But the partial and so total differentiations of \( (s_{\alpha,\rho})_\beta \) and \( (s'_{\alpha,\rho})_\beta \) necessarily accompany inhomogeneous Diophantine approximations unless \( \rho \in \alpha \mathbb{Z} + \mathbb{Z} \). Inhomogeneous Diophantine approximations are much more hard to manipulate than homogeneous ones.

In [10], the author exhibited that some pathological numbers in view of Diophantine approximation emerge in a rather natural situation. He showed that \( \Delta \) is not differentiable at values extremely well approximable by rationals, while differentiable at the other irrational values. The function \( \Xi \) also enjoys a similar property. We prove that if \( \alpha \) is not so extremely well approximable by rationals compared to \( \beta \), then \( \Xi \) is shown to be totally differentiable at \( (\alpha, \beta) \). It is worthwhile to mention here that the Minkowski \( ?(x) \) function is similar in spirit to \( \Xi \) and \( \Delta \) in a sense that Diophantine property determines differentiability [19, 6]. On the other hand, Beaver and Garrity [1] extended the Minkowski \( ?(x) \) function to a two-variable function. They showed that it is singular in a different sense.

Out of the values of \( \Xi \), we also find some algebraically independent numbers. More precisely, we will find some subset of the domain of \( \Xi \), any finite
number of values from which are algebraically independent over the field of rationals. This is a direct consequence of Masser’s results on the Hecke-Mahler series [17].

2. Diophantine preliminaries.

Let us write \( \|t\| \) for the distance between \( t \) and the nearest integer. To get the partial derivative of \( \Xi \) with respect to \( \alpha \), we need very special kind of Diophantine property of \( \alpha \). For any irrational \( \alpha \), we define the \textit{irrationality base} of \( \alpha \) by

\[
\theta(\alpha) := \sup \{ \lambda : \liminf_{q \to \infty} \frac{\lambda q}{q} \|q\alpha\| = 0 \}.
\]

This number was firstly coined by Sondow [21] and later pivotally exploited in [8, 10]. Recall that the usual irrationality measure of \( \alpha \) is given by

\[
\mu(\alpha) := \sup \{ \nu : \liminf_{q \to \infty} q^{\nu-1} \|q\alpha\| = 0 \}.
\]

This number is often called the \textit{irrationality exponent} of \( \alpha \). The definitions above ensure the following:

1. Every Diophantine number \( \alpha \) (i.e., \( \mu(\alpha) < \infty \)) has the irrationality base 1.
2. If \( \theta(\alpha) > 1 \), then \( \alpha \) is a Liouville number.

But there are Liouville numbers whose irrationality bases are equal to 1. In view of Jarník’s theorem [9], the numbers \( \alpha \) having \( \theta(\alpha) > 1 \) are very rare. The set of such numbers has Hausdorff dimension zero.

The next identity makes it possible to compute the irrationality base.

**Proposition 2.1** ([21, 10]). Let \( \alpha \) be irrational, and \( p_n/q_n \) be its \( n \)'th convergent. Then

\[
\log \theta(\alpha) = -\liminf_{n \to \infty} \frac{\log \|p_n\alpha\|}{q_n} = \limsup_{n \to \infty} \frac{\log q_{n+1}}{q_n}.
\]

The following properties of the irrationality base will play a key role when differentiating \( \Xi \).

**Lemma 2.2.** For irrational \( \alpha_0 \), let

\[
\delta_N := \min \left\{ \frac{\|\alpha_0 n\|}{n} : 1 \leq n \leq N \right\}.
\]

(1)
(a) If \(\theta(\alpha_0) < \lambda\), then there exists a positive integer \(N_0\) for which

\[
\bigcup_{n=N_0}^{\infty} (\lambda^{-n}, \delta_n) = (0, \delta_{N_0}).
\]

In other words, two consecutive intervals in the union have a nonempty intersection.

(b) If \(1 < \lambda < \theta(\alpha_0)\), then a set

\[
Q = \{n \in \mathbb{N} : \delta_n \leq \frac{\|\alpha_0n\|}{n} < \lambda^{-n}\}
\]

contains infinitely many denominators of the convergents of \(\alpha_0\).

Proof. See [10, Theorem 4.12].

3. Calculus on \(\Xi\).

We begin with a simple observation.

Lemma 3.1. As functions of \(\alpha\) from \([0, \infty)\) into \(\mathbb{N}\) both \(s_{\alpha,0}\) and \(s'_{\alpha,0}\) are continuous at every irrational. At rationals, \(s_{\alpha,0}\) (resp. \(s'_{\alpha,0}\)) is continuous from the right (resp. left) but not from the left (resp. right).

Proof. The claim follows from the corresponding continuity of the floor \(\lfloor \cdot \rfloor\) and the ceiling \(\lceil \cdot \rceil\) functions.

From the definition it follows that the function \(\Xi(\alpha, \beta)\) is continuous in \(\beta\) for any fixed \(\alpha\). For any fixed \(\beta\), Lemma 3.1 shows that \(\Xi(\alpha, \beta)\) is continuous at irrationals, and left-continuous but not right-continuous at rationals. The next proposition tells us that there are no other obstacles for the two-variable function \(\Xi(\alpha, \beta)\) to be continuous.

Proposition 3.2. The function \(\Xi(\alpha, \beta)\) is continuous at \((\alpha_0, \beta_0)\) if and only if \(\alpha_0\) is irrational.

Proof. For \(\alpha_0\) rational, \(\Xi(\alpha, \beta)\) is clearly discontinuous at \((\alpha_0, \beta_0)\). Suppose \(\alpha_0\) is irrational. Given \(\varepsilon > 0\), we will find \(\delta > 0\) so that

\[
\sqrt{(\alpha - \alpha_0)^2 + (\beta - \beta_0)^2} < \delta \text{ implies } |\Xi(\alpha, \beta) - \Xi(\alpha_0, \beta_0)| < \varepsilon.
\]
For $\delta > 0$ sufficiently small, we may assume that $[\alpha] = [\alpha_0] =: d$ and $\beta, \beta_0 \geq m$ for some constants $d \geq 1$ and $m > 1$. First, let

$$\delta' := \frac{(m - 1)^2 \varepsilon}{2d}.$$ 

Second, choose an integer $N > 0$ for which

$$\frac{d}{(m - 1)m^N} < \frac{\varepsilon}{4}.$$ 

And then set $\delta_N$ as in (1), which never vanishes since $\alpha_0$ is irrational. Then

$$\sqrt{(\alpha - \alpha_0)^2 + (\beta - \beta_0)^2} < \delta := \min\{\delta', \delta_N\}$$

implies that $|\beta - \beta_0| < \frac{(m-1)^2 \varepsilon}{2d}$ and that both $s'_{\alpha,0}$ and $s'_{\alpha_0,0}$ have a common prefix of length $N$. Let $\Xi(\alpha_0, \beta_0) = \sum_{i=1}^{\infty} \frac{a_i}{\beta_0^i}$ and $\Xi(\alpha, \beta) = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$, where $a_i = b_i$ for $1 \leq i \leq N$. The following inequality draws the conclusion:

$$|\Xi(\alpha, \beta) - \Xi(\alpha_0, \beta_0)| \leq \left| \sum_{i=1}^{N} a_i \left( \frac{1}{\beta_i} - \frac{1}{\beta_0^i} \right) \right| + \left| \sum_{i=N+1}^{\infty} a_i \frac{1}{\beta_0^i} \right| + \left| \sum_{i=N+1}^{\infty} b_i \frac{1}{\beta^i} \right|$$

$$\leq d \cdot \frac{|\beta - \beta_0|}{|\beta_0|} \cdot \left| 1 + \frac{1}{\beta + \beta_0} + \frac{1}{\beta^2 + \beta_0^2} + \cdots \right|$$

$$+ d \cdot \frac{1}{(\beta - 1)\beta^N} + \frac{1}{(\beta_0 - 1)\beta_0^N}$$

$$\leq \frac{d \cdot |\beta - \beta_0|}{m^2} \cdot \frac{m^2}{(m - 1)^2} + \frac{2d}{(m - 1)m^N} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Let us now discuss partial and total differentiability of $\Xi(\alpha, \beta)$. To check differentiability, we should appeal to more accurate estimation. It will turn out that the estimation strongly rests upon some Diophantine property of $\alpha$. This nature is inherited from the well-known connection between mechanical words and Diophantine property of their slopes [14, 2].

We consider partial differentiability first. On the one hand, $\Xi(\alpha, \beta) = (s'_{\alpha,0})_\beta$ is smooth as a function of $\beta$ only. On the other hand, Theorem 3.4 enables us to differentiate $\Xi(\alpha, \beta)$ with respect to $\alpha$ except on a set of Hausdorff dimension zero. In the proof, we need to compare mechanical words lexicographically. The next lemma is well known. See, e.g., [14, 2].

$$\Xi(\alpha, \beta) = (s'_{\alpha,0})_\beta$$
Lemma 3.3. Let $0 < \rho < 1$ and suppose that $\alpha > 0$ is not an integer. Then we have lexicographically
\[ s_{\alpha,0} \leq s'_{\alpha,\rho} \leq s_{\alpha,\rho} \leq s'_{\alpha,0}. \]

The partial differentiability of $\Xi$ behaves in a similar fashion to that of $\Delta$. So the corresponding proof for differentiation of $\Delta$ in [10] also works here, but with careful modifications. One should keep in mind that $s'_{\alpha,0}$ below is not necessarily a $\beta_0$-expansion obtained by greedy algorithm (cf. [20]).

Theorem 3.4. Let $(\alpha_0, \beta_0)$ be fixed, and suppose that $\alpha_0$ is irrational.

(a) If $\theta(\alpha_0) < \beta_0$, then $\frac{\partial \Xi}{\partial \alpha}(\alpha_0, \beta_0) = 0$.
(b) If $\theta(\alpha_0) > \beta_0$, then $\Xi(\alpha, \beta)$ is not partially differentiable with respect to $\alpha$ at $(\alpha_0, \beta_0)$.

Proof. (a) Define $\delta_N$ as in (1), and let $\lambda$ be such that $\theta(\alpha_0) < \lambda < \beta_0$. For any sequence $(\alpha_k)_{k \geq 1}$ such that each $\alpha_k - \alpha_0$ never vanishes and that $\lim_{k \to \infty} \alpha_k = \alpha_0$, it suffices to show that
\[
\lim_{k \to \infty} \frac{|\Xi(\alpha_k, \beta_0) - \Xi(\alpha_0, \beta_0)|}{|\alpha_k - \alpha_0|} = 0.
\]

By Lemma 2.2 (a), the sequence $|\alpha_k - \alpha_0|$ eventually belongs to the union $\bigcup_{n=N_0}^{\infty} (\lambda^{-n}, \delta_n) = (0, \delta_{N_0})$ for some $N_0$. Thus, for $k$ sufficiently large,
\[
\lambda^{-n} < |\alpha_k - \alpha_0| < \delta_n
\]
and furthermore $n \to \infty$ as $k \to \infty$. For any $\alpha$ satisfying $\lambda^{-n} < |\alpha - \alpha_0| < \delta_n$, the word $s'_{\alpha,0}$ has a common prefix of length $n$ with $s'_{\alpha_0,0}$. We may assume as before that $[\alpha] = [\alpha_0] =: d$. Consequently, one derives the following inequality for $\Xi(\alpha_0, \beta_0) = \sum_{i=1}^{\infty} \frac{a_i}{\beta_0^i}$ and $\Xi(\alpha, \beta_0) = \sum_{i=1}^{\infty} \frac{b_i}{\beta_0^i}$:
\[
|\Xi(\alpha, \beta_0) - \Xi(\alpha_0, \beta_0)| \leq \left| \sum_{i=n+1}^{\infty} \frac{a_i}{\beta_0^i} \right| + \left| \sum_{i=n+1}^{\infty} \frac{b_i}{\beta_0^i} \right| \leq \frac{2d}{(\beta_0 - 1)\beta_0^n}.
\]

We conclude
\[
\frac{|\Xi(\alpha_k, \beta_0) - \Xi(\alpha_0, \beta_0)|}{|\alpha_k - \alpha_0|} \leq \frac{2d\lambda^n}{(\beta_0 - 1)\beta_0^n} \to 0 \text{ as } k \to \infty.
\]
(b) Let \( \lambda \) be such that \( \beta_0 < \lambda < \theta(\alpha_0) \). According to Lemma 2.2 (b), we suppose that every \( q_i < q_j < \cdots \) lies in \( Q \) and is the denominator of the convergent \( p_i/q_i \) of \( \alpha_0 \). Thanks to the best approximation property of convergents, one has

\[
\delta_q = \frac{\|\alpha_0 q_i\|}{q_i} < \lambda^{-q_i} \quad \text{and} \quad \delta_q < \delta_{q-1}.
\]

For each \( i \geq 1 \), we set

\[
\alpha_i := \begin{cases} 
\alpha_0 - \delta_q & \text{if } \|q_i \alpha_0\| = q_i \alpha_0 - p_i, \\
\alpha_0 + \tau_i & \text{if } \|q_i \alpha_0\| = p_i - q_i \alpha_0,
\end{cases}
\]

where \( \tau_i \) is chosen so that \( \delta_q < \tau_i \leq \min \{ \frac{\delta_q + \delta_{q-1}}{2}, 2\delta_q \} \) and that \( \alpha_0 + \tau_i \) is irrational. Consequently, one has

\[
\lim_{i \to \infty} \alpha_i = \alpha_0 \quad \text{and} \quad |\alpha_i - \alpha_0| \leq 2\delta_q < \frac{2}{\lambda^q}.
\]

Moreover, suppose \( s'_{\alpha_0,0} = a_1 a_2 \cdots \) and \( s'_{\alpha_i,0} = b_1 b_2 \cdots \). Then \( a_1 \cdots a_{q-1} = b_1 \cdots b_{q-1} \), but

\[
b_q = \begin{cases} 
\alpha_i - 1 & \text{if } \|q_i \alpha_0\| = q_i \alpha_0 - p_i, \\
\alpha_i + 1 & \text{if } \|q_i \alpha_0\| = p_i - q_i \alpha_0.
\end{cases}
\]

Note that if \( \|q_i \alpha_0\| = q_i \alpha_0 - p_i \) then \( s'_{\alpha_i,0} = (b_1 b_2 \cdots b_q)^\omega \) is periodic. In the case of \( \|q_i \alpha_0\| = q_i \alpha_0 - p_i \), one deduces from Lemma 3.3 that

\[
|\Xi(\alpha_i, \beta_0) - \Xi(\alpha_0, \beta_0)| = (s'_{\alpha_0,0})_{\beta_0} - (s'_{\alpha_i,0})_{\beta_0}
\]

\[
= \left( \frac{a_1}{\beta_0^{q_1}} + \frac{a_1+a_2}{\beta_0^{q_1+1}} + \frac{a_1+a_2+2}{\beta_0^{q_1+2}} + \cdots \right) - \left( \frac{a_1-1}{\beta_0^{q_1}} + \frac{b_1+1}{\beta_0^{q_1+1}} + \frac{b_1+2}{\beta_0^{q_1+2}} + \cdots \right)
\]

\[
= \frac{1}{\beta_0^{q_1} - \beta_0^{q_1+1}} + \frac{1}{\beta_0^{q_1+1}} \left( (s'_{\alpha_0,0})_{\beta_0} - (s'_{\alpha_i,0})_{\beta_0} \right)
\]

\[
\geq \left( 1 - \frac{1}{\beta_0} \right) \left( \frac{1}{\beta_0^{q_1}} + \frac{1}{\beta_0^{q_2}} + \cdots \right) = \left( 1 - \frac{1}{\beta_0} \right) \frac{1}{\beta_0^{q_1}} \geq \left( 1 - \frac{1}{\beta_0} \right) \frac{1}{\beta_0^{q_1}}.
\]
Adapting a similar reasoning to the case of $\|q_i\alpha_0\| = p_i - q_i\alpha_0$, we find that

$$
|\Xi(\alpha_i, \beta_0) - \Xi(\alpha_0, \beta_0)| = (s'_{\alpha_i, 0})_{\beta_0} - (s'_{\alpha_0, 0})_{\beta_0}
$$

$$
= \left(\frac{a_{q_i} + 1}{\beta_0^{q_i}} + \frac{b_{q_i+1}}{\beta_0^{q_i+1}} + \frac{b_{q_i+2}}{\beta_0^{q_i+2}} + \cdots\right) - \left(\frac{a_{q_0} + 1}{\beta_0^{q_0}} + \frac{a_{q_0+1}}{\beta_0^{q_0+1}} + \frac{a_{q_0+2}}{\beta_0^{q_0+2}} + \cdots\right)
$$

$$
= \frac{1}{\beta_0^{q_i}} + \frac{1}{\beta_0^{q_i+1}} \left((s'_{\alpha_i, 0})_{\beta_0} - (s'_{\alpha_0, 0})_{\beta_0}\right) \geq \frac{1}{\beta_0^{q_i}} + \frac{1}{\beta_0^{q_i+1}} \left((s_{\alpha_i})_{\beta_0} - (s'_{\alpha_0, 0})_{\beta_0}\right)
$$

$$
= \frac{1}{\beta_0^{q_i}} - \frac{1}{\beta_0^{q_i+1}} + \frac{1}{\beta_0^{q_i+1}} \left((s'_{\alpha_i, 0})_{\beta_0} - (s'_{\alpha_0, 0})_{\beta_0}\right) \quad (\because \alpha_i \text{ is irrational})
$$

$$
\geq \left(1 - \frac{1}{\beta_0}\right) \left(\frac{1}{\beta_0^{q_i}} + \frac{1}{\beta_0^{q_i+1}} + \cdots\right) \geq \left(1 - \frac{1}{\beta_0}\right) \frac{1}{\beta_0^{q_i}} \geq \left(1 - \frac{1}{\beta_0}\right) \frac{1}{\beta_0^{q_i}}.
$$

We finally get

$$
\frac{|\Xi(\alpha_i, \beta_0) - \Xi(\alpha_0, \beta_0)|}{|\alpha_i - \alpha_0|} \geq \left(1 - \frac{1}{\beta_0}\right) \frac{\lambda^{q_i}}{2\beta_0^{q_i}} \to \infty \quad \text{as} \quad i \to \infty.
$$

Currently, if $\theta(\alpha_0) = \beta_0$ accidentally, then we cannot decide whether or not $\Xi(\alpha, \beta)$ is partially differentiable with respect to $\alpha$ at $(\alpha_0, \beta_0)$. Either proof of Theorem 3.4 does not work for this case.

**Example 1.** Suppose $a_1 = 10$ and $a_n = 10^{a_{n-1}}$ for $n \geq 2$. Let

$$
\alpha_0 = \sum_{k=1}^{\infty} \frac{1}{a_k} = \frac{1}{10} + \frac{1}{10^{10}} + \frac{1}{10^{10^{10}}} + \cdots.
$$

Then $\theta(\alpha_0) = 10$. So we do not know whether $\Xi(\alpha, \beta)$ is partially differentiable with respect to $\alpha$ at $(\alpha_0, 10)$.

**Proof.** Let $(p_n/q_n)_{n \geq 1}$ be the sequence of the $n$’th convergents of $\alpha_0$, and let

$$
r_m \equiv \sum_{k=1}^{m} \frac{1}{a_k}
$$

be fractions with $\gcd(r_m, s_m) = 1$. One notes that $s_m = a_m$ and

$$
\alpha_0 - \frac{r_m}{s_m} < \frac{10}{9 \cdot a_{m+1}} < \frac{1}{2s_m^2},
$$
which in turn implies that \( (r_m/s_m)_{m \geq 1} \) is a subsequence of \( (p_n/q_n)_{n \geq 1} \). If 
\((q_n)_i \geq 1\) satisfies
\[
\lim_{i \to \infty} \frac{\log q_{n+1}}{q_n} = \log \theta(\alpha_0),
\]
then we can pick a (sub)sequence \((s_m)_i \geq 1\) such that \(s_m \leq q_n\) and \(q_{n+1} \leq s_{m+1}\) hold. Here, \((s_m)_i \geq 1\) is not a usual subsequence of \((s_m)_{m \geq 1}\). Some \(m_i\) may be equal to \(m_i+1\). Now one derives
\[
\log \theta(\alpha_0) = \lim_{i \to \infty} \frac{\log q_{n+1}}{q_n} \leq \lim_{i \to \infty} \frac{\log s_{m+1}}{s_m} = \lim_{i \to \infty} \frac{s_{m_i} \log 10}{s_m} = \log 10.
\]
Since \((s_m)_{m \geq 1}\) is a subsequence of \((q_n)_{n \geq 1}\), Proposition 2.1 implies
\[
\log \theta(\alpha_0) \geq -\liminf_{m \to \infty} \frac{\log \|s_m \alpha_0\|}{s_m}.
\]
But we readily note
\[
\|s_m \alpha_0\| = a_m \sum_{k=m+1}^{\infty} \frac{1}{a_k} \leq \frac{2a_m}{a_{m+1}}.
\]
Thus we find
\[
\log \theta(\alpha_0) \geq -\liminf_{m \to \infty} \frac{\log (2s_m/s_{m+1})}{s_m} = -\liminf_{m \to \infty} \frac{\log 2 + \log s_m - \log s_{m+1}}{s_m} = \limsup_{m \to \infty} \frac{s_m \log 10}{s_m} = \log 10.
\]
\(\square\)

Theorem 3.4 tells us that except on a set of Hausdorff dimension zero, \(\Xi(\alpha, \beta)\) is partially differentiable with respect to both \(\alpha\) and \(\beta\) simultaneously. More precisely, if \(\theta(\alpha_0) < \beta_0\) and if \(s'_{\alpha_0,0} = a_1 a_2 \cdots\), then the gradient of \(\Xi\) at \((\alpha_0, \beta_0)\) is given by
\[
\text{grad } \Xi(\alpha_0, \beta_0) = \left(0, -\sum_{i=1}^{\infty} \frac{i a_i}{\beta_0^{i+1}}\right).
\]
Since \(\frac{\partial \Xi}{\partial \alpha}\) is far from being continuous, the existence of the gradient at \((\alpha_0, \beta_0)\) never guarantees that \(\Xi\) is totally differentiable. We will see however that \(\Xi\)

10
is indeed totally differentiable at \((\alpha_0, \beta_0)\) whenever \(\theta(\alpha_0) < \beta_0\). Recall that a function \(f : \mathbb{R}^2 \to \mathbb{R}\) is totally differentiable at \((x_0, y_0)\) if the gradient of \(f\) exists at \((x_0, y_0)\), and if
\[
\lim_{(x,y) \to (x_0,y_0)} \frac{|f(x,y) - f(x_0,y_0) - (x-x_0, y-y_0) \cdot \nabla f(x_0,y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.
\]

**Theorem 3.5.** Let \(\alpha_0\) be irrational. If \(\theta(\alpha_0) < \beta_0\), then \(\Xi(\alpha, \beta)\) is totally differentiable at \((\alpha_0, \beta_0)\).

**Proof.** Suppose \(\theta(\alpha_0) < \beta_0\). We will show that, for any sequence \((\alpha_k, \beta_k) \neq (\alpha_0, \beta_0)\) converging to \((\alpha_0, \beta_0)\),
\[
\lim_{k \to \infty} \frac{|\Xi(\alpha_k, \beta_k) - \Xi(\alpha_0, \beta_0) + (\beta_k - \beta_0) \sum_{i=1}^{\infty} \frac{ia_i}{\beta_0^i+1}|}{\sqrt{(\alpha_k - \alpha_0)^2 + (\beta_k - \beta_0)^2}} = 0.
\]

For sufficiently large \(k\), we may assume that \([\alpha_0] = [\alpha_k] =: d\) and \(\beta_0, \beta_k \in [m, M]\) for some constants \(d, m, M\). Furthermore, we can also assume that \(\theta(\alpha_0) < m^2/M\). Pick a real \(\lambda\) satisfying
\[
\theta(\alpha_0) < \lambda < m^2/M \leq m.
\]

Let \(\delta_n\) be as in (1). Then Lemma 2.2 guarantees that, for all \(k\) sufficiently large,
\[
\lambda^{-n} < \sqrt{(\alpha_k - \alpha_0)^2 + (\beta_k - \beta_0)^2} < \delta_n
\]
and \(n \to \infty\) as \(k \to \infty\). Let \(s'_{\alpha_0,0} = a_1a_2\cdots\) and \(s'_{\alpha_k,0} = b_1b_2\cdots\). Since \(|\alpha_k - \alpha_0| < \delta_n\) and thus \(a_1 = b_1, \ldots, a_n = b_n\), we rearrange terms as follows:
\[
\begin{align*}
\Xi(\alpha_k, \beta_k) - \Xi(\alpha_0, \beta_0) + (\beta_k - \beta_0) \sum_{i=1}^{\infty} \frac{ia_i}{\beta_0^i+1} & \\
\leq \left| (\beta_k - \beta_0) \sum_{i=n+1}^{\infty} \frac{ia_i}{\beta_0^i+1} \right| & \quad (2) \\
+ \left| \sum_{i=n+1}^{\infty} \frac{b_i}{\beta_k^i} - \sum_{i=n+1}^{\infty} \frac{a_i}{\beta_0^i} \right| & \quad (3) \\
+ \left| \sum_{i=1}^{n} a_i \left( \frac{1}{\beta_k^i} - \frac{1}{\beta_0^i} \right) + (\beta_k - \beta_0) \sum_{i=1}^{n} \frac{ia_i}{\beta_0^i+1} \right| & \quad (4)
\end{align*}
\]
Now it suffices to show that each term divided by $\sqrt{(\alpha_k - \alpha_0)^2 + (\beta_k - \beta_0)^2}$ converges to zero as $k \to \infty$.

First, because the series $\sum_{i=1}^{\infty} \frac{ia_i}{\beta_0^{i+1}}$ converges, the sequence $\sum_{i=n+1}^{\infty} \frac{ia_i}{\beta_0^{i+1}}$ converges to zero as $n$ tends to infinity, and so does

$$\lim_{k \to \infty} \left| \frac{(\beta_k - \beta_0) \sum_{i=n+1}^{\infty} \frac{ia_i}{\beta_0^{i+1}}}{\sqrt{(\alpha_k - \alpha_0)^2 + (\beta_k - \beta_0)^2}} \right| = 0.$$

As for the term (3), one notes that

$$\left| \sum_{i=n+1}^{\infty} \frac{b_i}{\beta_k^i} - \sum_{i=n+1}^{\infty} \frac{a_i}{\beta_0^i} \right| \leq \frac{2d}{(m-1)m^n}.$$

Therefore,

$$\frac{\sum_{i=n+1}^{\infty} \frac{b_i}{\beta_k^i} - \sum_{i=n+1}^{\infty} \frac{a_i}{\beta_0^i}}{\sqrt{(\alpha_k - \alpha_0)^2 + (\beta_k - \beta_0)^2}} \leq \frac{2d\lambda^n}{(m-1)m^n} \to 0 \text{ as } k \to \infty.$$

Next, we turn to the term (4). One derives

$$\left| \sum_{i=1}^{n} a_i \left( \frac{1}{\beta_k^i} - \frac{1}{\beta_0^i} \right) + (\beta_k - \beta_0) \sum_{i=1}^{n} \frac{ia_i}{\beta_0^{i+1}} \right|$$

$$\leq \frac{d|\beta_k - \beta_0|}{m} \sum_{i=1}^{n} \frac{|i\beta_k^i - \beta_0(i\beta_k^{i-1} + \beta_k^{i-2} - \beta_0 + \cdots + \beta_0^{i-1})|}{m^{2i}}$$

$$= \frac{d|\beta_k - \beta_0|}{m} \sum_{i=1}^{n} \frac{|\beta_k^{i-1}(\beta_k - \beta_0) + \beta_k^{i-2}(\beta_k^2 - \beta_0^2) + \cdots + (\beta_k^i - \beta_0^i)|}{m^{2i}}$$

$$= \frac{d|\beta_k - \beta_0|^2}{m} \sum_{i=1}^{n} \frac{|\beta_k^{i-1} + \beta_k^{i-2}(\beta_k + \beta_0) + \cdots + (\beta_k^{i-1} + \cdots + \beta_0^{i-1})|}{m^{2i}}$$

$$\leq \frac{d|\beta_k - \beta_0|^2}{m} \sum_{i=1}^{n} \frac{M^{i-1} + 2M^{i-1} + \cdots + iM^{i-1}}{m^{2i}}$$

$$\leq \frac{d|\beta_k - \beta_0|^2}{2Mm} \sum_{i=1}^{n} i(i + 1) \left( \frac{M}{m^2} \right)^i.$$
A routine calculation leads us to
\[ \sum_{i=1}^{n} i(i+1) \left( \frac{M}{m^2} \right)^i = \left( \frac{M}{m^2} \right)^n p(n) + c_0, \]
where \( p(n) \) is a quadratic polynomial and \( c_0 \) is a constant. For \( c_1 = \frac{d}{2Mm} \), we finally conclude

\[ \left| \sum_{i=1}^{n} a_i \left( \frac{1}{\beta_k} - \frac{1}{\beta_0} \right) + (\beta_k - \beta_0) \sum_{i=1}^{n} \frac{ia_i}{\beta_0^{i+1}} \right| \]
\[ \leq c_1 |\beta_k - \beta_0|^2 \left( \frac{M\lambda}{m^2} \right)^n p(n) + c_0 |\beta_k - \beta_0| \to 0 \text{ as } k \to \infty, \]
since \( M\lambda/m^2 < 1 \).

\[ \square \]

4. Arithmetic on \( \Xi \).

In this section, we obtain transcendence and algebraic independence results of \( \Xi \)-values. These are all simple consequences of known results on the Hecke-Mahler series, which is also called the Böhmer-Mahler series in the literature [22].

The next theorem is due to [16, 15], and extends the previous work [5].

**Theorem 4.1.** Let \( \xi = \Xi(\alpha, \beta) \).

(a) If \( \alpha \) is rational, then \( \xi \in \mathbb{Q}(\beta) \).
(b) If \( \alpha \) is irrational, then either \( \beta \) or \( \xi \) is transcendental.

The Hecke-Mahler series is a power series defined by

\[ f(\alpha, z) = \sum_{n=1}^{\infty} \lfloor n\alpha \rfloor z^n, \]

where \( \alpha \) is real and \( z \) is complex with \( |z| < 1 \). Masser [17] proved the following two algebraic independence properties of the Hecke-Mahler series.
Proposition 4.2. Let $\alpha$ be a quadratic irrational and $\zeta_1, \ldots, \zeta_n$ be nonzero algebraic numbers with $|\zeta_1|, \ldots, |\zeta_n| < 1$. Then $f(\alpha, \zeta_1), \ldots, f(\alpha, \zeta_n)$ are algebraically independent over $\mathbb{Q}$ if and only if all of $\zeta_1, \ldots, \zeta_n$ are distinct.

Proposition 4.3. Let $\alpha_1, \ldots, \alpha_n$ be quadratic irrationals and $\zeta$ be a nonzero algebraic number with $|\zeta| < 1$. Then $f(\alpha_1, \zeta), \ldots, f(\alpha_n, \zeta)$ are algebraically independent over $\mathbb{Q}$ if and only if all of $\pm \alpha_1, \ldots, \pm \alpha_n$ are distinct modulo the rational integers.

Let $\alpha > 0$ be irrational and $\beta > 1$. Then we observe that

\[
\Xi(\alpha, \beta) = \sum_{n=0}^{\infty} \frac{\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor}{\beta^{n+1}} = 1 + \sum_{n=0}^{\infty} \frac{\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor}{\beta^{n+1}} = 1 + \left(1 - \frac{1}{\beta}\right) f(\alpha, \beta^{-1}).
\]

This identity ensures the algebraic independence of the following $\Xi$-values.

**Theorem 4.4.** Let $\alpha > 0$ be a quadratic irrational and $\xi_i = \Xi(\alpha, \beta_i)$, where all of $\beta_1, \ldots, \beta_n$ are distinct algebraic numbers greater than 1. Then $\xi_1, \ldots, \xi_n$ are algebraically independent over $\mathbb{Q}$.

**Theorem 4.5.** Let $\beta > 1$ be an algebraic number and $\xi_i = \Xi(\alpha_i, \beta)$, where $\alpha_1, \ldots, \alpha_n$ are positive quadratic irrationals such that all of $\pm \alpha_1, \ldots, \pm \alpha_n$ are distinct modulo the rational integers. Then $\xi_1, \ldots, \xi_n$ are algebraically independent over $\mathbb{Q}$.

Two theorems immediately follow from the next lemma. This is elementary but hard to find in the literature.

**Lemma 4.6.** Let $\gamma_1, \ldots, \gamma_n$ be algebraically independent over $\mathbb{Q}$ and $\eta$ be a nonzero algebraic number. Then

(a) $\eta \gamma_1, \gamma_2, \ldots, \gamma_n$ are algebraically independent over $\mathbb{Q}$,

(b) $\eta + \gamma_1, \gamma_2, \ldots, \gamma_n$ are algebraically independent over $\mathbb{Q}$.

**Proof.** We prove (a). The proof for (b) is similar. Suppose that $\eta \gamma_1, \gamma_2, \ldots, \gamma_n$ are algebraically dependent over $\mathbb{Q}$. Let $f(\eta \gamma_1, \gamma_2, \ldots, \gamma_n) = 0$ for some $f \in \mathbb{Q}[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are indeterminates. Note that $f \notin \mathbb{Q}[x_2, \ldots, x_n]$, i.e., $\eta \gamma_1$ divides some monomial term of $f(\eta \gamma_1, \gamma_2, \ldots, \gamma_n)$. 

Otherwise, $\gamma_2, \ldots, \gamma_n$ are algebraically dependent over $\mathbb{Q}$. So $\eta \gamma_1$ is algebraic over $\mathbb{Q}(\gamma_2, \ldots, \gamma_n)$. Since $\eta$ is algebraic over $\mathbb{Q}$ and hence over $\mathbb{Q}(\gamma_2, \ldots, \gamma_n)$, we find that $\eta \gamma_1 \cdot \frac{1}{\eta} = \gamma_1$ is algebraic over $\mathbb{Q}(\gamma_2, \ldots, \gamma_n)$. In other words, there is $g \in \mathbb{Q}[\gamma_2, \ldots, \gamma_n][x_1]$ such that $g(\gamma_1) = 0$, which implies that $\gamma_1, \gamma_2, \ldots, \gamma_n$ are algebraically dependent over $\mathbb{Q}$. \hfill \Box

Finally, we report on additional arithmetic properties of $\Xi$-values, which are far from being exhaustive. Lemma 4.6 translates, without difficulty, each reference below into our corresponding result.

**Theorem 4.7.** (a) Let $\alpha > 0$ have unbounded partial quotients. If $\beta \geq 2$ is an integer, then $\Xi(\alpha, \beta)$ is a Liouville number. ([3])

(b) Let $\alpha > 0$ be a quadratic irrational. If $\beta > 1$ is an algebraic number, then $\Xi(\alpha, \beta)$ is transcendental but not a Liouville number. ([16])

(c) Let $\alpha > 0$ be irrational. If $\beta_1, \ldots, \beta_n > 1$ are multiplicatively independent algebraic numbers, then $\Xi(\alpha, \beta_1), \ldots, \Xi(\alpha, \beta_n)$ are algebraically independent over $\mathbb{Q}$. ([7])

(d) Let $\alpha > 0$ have unbounded partial quotients. If $\beta_1, \ldots, \beta_n > 1$ are algebraic numbers such that $\beta_i/\beta_j$ is never a root of unity for $i \neq j$, then $\Xi(\alpha, \beta_1), \ldots, \Xi(\alpha, \beta_n)$ are algebraically independent over $\mathbb{Q}$. ([18])

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15


